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Polarization of the magnetized scalar and spinor vacua

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Abstract. The vacuum polarization tensor of the magnetized boson system, taken within the random phase or one-loop approximation, is presented in three representation forms: the Landau, the proper-time, and the dispersion sum representation. Detailed numerical investigations into many aspects of the physical properties of both the bosonic and fermionic vacua are presented on the basis of these analytical results. From the static and uniform limit of the polarization scalars it is found that the boson vacuum state enhances small electrostatic fields parallel to the external field as well in the transverse direction contrary to the screening behaviour of the fermion vacuum, and all the scalars have a weak logarithmic growth in strong fields. Also contrary to the fermionic behaviour, for wavevectors parallel to the field, the longitudinal dielectric function does not exhibit any singularities at the pair production thresholds. Furthermore the 'massive longitudinal photon' mode found for the magnetized fermion vacuum does not exist in the bosonic equivalent. Like the spin-$\frac{1}{2}$ case the dispersion solutions show that the purely transverse 'photon' mode (3) acquires mass and is channelled along the field lines, thus manifesting the mixed state of a photon and a boson–antiboson quasibound state. However, the other mode (2), which is a combined longitudinal-transverse mode does not deviate significantly from the free-space dispersion law and its inverse lifetime has no singularities at the pair thresholds.

1. Introduction

Interest in the nature of the magnetized vacuum has been aroused by some studies of photon propagation in a strong external magnetic field that have demonstrated the rather unique behaviour in that the group velocity becomes aligned along the local field [SH72, SH75, SH84], that is to say that the photon no longer propagates rectilinearly. The explanation of these dispersion solutions in terms of not a pure photon but of a quasiparticle which acquires mass through the mixing with bound states of positronium has astrophysical implications [SH82, SH84, SH85, SH86]. The effect of this is to inhibit pair creation processes in a curved magnetic field, and to put in doubt the key assumption in the electron–positron pulsar models that are currently favoured. This work seeks to repeat many of the earlier fermion calculations except with zero-spin charged bosons, that is with a different vacuum state, and to find exact numerical solution modes for photon propagation and lifetimes in both cases. In addition it is intended to physically describe the bosonic vacuum because this needs to be incorporated into any relativistic description of the magnetized zero-spin pair plasma.

The first detailed exposition of the structure of the tensor and the eigenmodes that propagate in the vacuum in the presence of a constant but otherwise arbitrary external electromagnetic field appeared in Batalin and Shabad [BT71] which not only treated the external field exactly, but was valid for all orders in the radiation field. It should also be noted that in this work no approximations were made with respect to the photon's momentum, i.e. it was not assumed to lie on the light cone, or approximate mass shell. In this fundamental paper an explicit calculation of the tensor was made in the one-loop approximation using the Schwinger electron propagator in such a general external electromagnetic field, although no specific solutions were investigated at this point. The original paper [BT71] was followed by another [SH75] which treated the special case of a pure magnetic field, and in this a number of new ideas were present. Firstly the analyticity of the three polarization scalars was discussed in terms of the complex photon momentum scalars and the Landau sum, or spectral representation was developed from the proper-time one. This was the first paper to treat the photon dispersion solutions in full generality. Furthermore the qualitative aspects of all the possible solutions were sketched out and in particular analytical solutions were developed in the region of the pair production thresholds, where it was first noted that the group velocity of the photons aligns itself with the magnetic field direction. Part of this phenonemon is that $k_{\|}^2 - \omega^2$ tended to asymptote approximately towards the pair production thresholds, $-\left\{\sqrt{m^2c^4 + 2e\hbar c^2 Bn} + \sqrt{m^2c^4 + 2e\hbar c^2 Bn'}\right\}^2$, the squares of the energy of the Landau levels of the emergent electron and positron, and this was attributed to being the manifestation of the mixing of photons and quasibound states of electron–positron pairs, positronium.

A complementary development to that of Shabad *et al* was made by Bakshi *et al* who developed a form for the polarization tensor in the uniform magnetic field based on three scalars, also without assuming the photons lay on the light cone, i.e. as in Shabad *et al.* In the first paper [CV74] they reported numerical investigations of a particular solution to dispersion relations. They found a solution of the longitudinal mode for the particular case when the perpendicular wave number was zero (which coincides with a transverse mode in this case) just above the first pair creation threshold, i.e $\omega \sim 2mc^2$ with $k_{\|} = 0$ also, but only for magnetic field strengths $B > 429B_\star (B_\star$ is defined later). This frequency solution increased monotonically with field strength and they also calculated its lifetime. Following on from this they investigated the static and uniform limits of the three vacuum scalars [BK75], which they numerically computed for a large range of field strengths. They found electric field enhancement in the transverse direction and one scalar grew asymptotically as B in large fields. The third paper [BK76] has a derivation of the polarization tensor based on the spectral representation of the electron Green function and discusses three renormalization schemes that can be used with the Landau sum representation of the scalars. The only restriction of the results is, as with the earlier work, is that all considerations are confined to a vanishing perpendicular photon wavenumber. It also contained the most extensive numerical work to date on the magnetized vacuum, and the results of [CV74] were confirmed. They dubbed this a 'massive longitudinal photon' and interpreted it as being the antiparallel spin $s = 0$ quasibound electron–positron pair which forms because of the electric field enhancement in the magnetized vacuum. Other authors have also made calculations of the vacuum polarization tensor, such as [BI75, MR76, MR77, SV85].

The plan of the work presented in this paper is the following. In the remainder of this section a number of key conventions and definitions are made, along with the pertinent general properties of the vacuum polarization tensor. In section 2 the proper-time representation of the boson tensor is found from the earlier random phase calculation in [WT88], while in section 3 the Landau sum or spectral representation is derived from this and its convergence properties discussed, and in section 4 a dispersion-sum representation is found for the vacuum tensor. In section 5 the anti-Hermitian parts of the tensor are presented as well as the inverse lifetimes of the propagating modes. The physical description of the vacua begins with the discussion of the static and uniform limits of the scalars both analytically and numerically in section 6—here one can find a discussion of the screened electrostatic potential of a test charge. This is generalized in section 7, which considers the behaviour of all the vacuum scalars with respect to the photon momenta on the basis of analytical and numerical calculations. Finally in section 8 numerical solutions are presented for the non-trivial dispersion solutions to all the photon modes and are discussed with some analytical approximations.

From the work of Batalin and Shabad [BT71], it has been clear that the vacuum polarization tensor in the presence of an uniform and static magnetic field can be characterized quite generally, that is to say non-perturbatively, by three scalars. Furthermore the polarization tensor is diagonal in the orthonormal basis $a^{(m)}$, $m = 1, 2, 3$ (see [WT88] for more details on the definitions adopted here) defined by

$$a^{(1)}{}_{\mu} = \frac{\left[k^2(F^2k)_\mu - k_\mu(kF^2k)\right]}{\left[k^2(kF^2k)(kF^{\star 2}k)\right]^{1/2}} \qquad a^{(1)} \cdot a^{(1)} = -1 \tag{1.1a}$$

$$a^{(2)}{}_{\mu} = -\frac{(F^\star k)_\mu}{\left[(kF^{\star 2}k)\right]^{1/2}} \qquad a^{(2)} \cdot a^{(2)} = -1 \tag{1.1b}$$

$$a^{(3)}{}_{\mu} = \frac{(Fk)_\mu}{\left[(kF^2k)\right]^{1/2}} \qquad a^{(3)} \cdot a^{(3)} = -1 \tag{1.1c}$$

where the symbolic notation is

$$(F^{\star m}F^n)_{\mu\nu} = F^{\star\alpha_1}_{\mu} F^{\star}_{\alpha_1}{}^{\alpha_2} \cdots F^{\star}_{\alpha_m}{}^{\alpha_{m+1}} F_{\alpha_{m+1}}{}^{\alpha_{m+2}} \cdots F_{\alpha_{m+n}\nu}$$

with $F_{\alpha\beta} \equiv A_{\alpha,\beta} - A_{\beta,\alpha}$ the electromagnetic tensor of the external field and its dual tensor, $F^\star_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. The 4-momentum of the photon is denoted by $\boldsymbol{k}$. The three non-zero scalars are then the $\mathcal{P}, \mathcal{S}$ and $\mathcal{T}$ that were defined for the magnetized plasma in [WT88] and in summary one can write the polarization tensor as

$$\Pi_{\mu\nu} = \mathcal{P}a^{(1)}_\mu a^{(1)}_\nu + \mathcal{S}a^{(2)}_\mu a^{(2)}_\nu + \mathcal{T}a^{(3)}_\mu a^{(3)}_\nu \ . \tag{1.2}$$

These scalars are in turn functions of only three variables: the magnetic field strength parameter $\beta = 2B/B_\star = 2b^2$ where $B_\star = m^2c^2/(2e\hbar)$ in SI units, and the parallel and perpendicular 'squared photon energies' $y = q_{\parallel}^2 - \Omega^2$ and $z = q_\perp^2 = \beta z$ (these two quantities should not be confused) respectively (in all that follows the dimensionless units $\Omega = \hbar\omega/mc^2$ and $\boldsymbol{q} = \hbar\boldsymbol{k}/mc$ or $\boldsymbol{\mu} = \sqrt{\hbar/eB}\boldsymbol{k}$ are used with $z = \frac{1}{2}\mu_\perp^2$). The wavevector $\boldsymbol{q}$ is confined to the x–z plane with a component perpendicular $q_\perp$ and a component parallel $q_{\parallel}$ to the external magnetic field, which is directed along the positive z axis. These conclusions are also independent of the details of the particular dynamics of the theory or the order to which a perturbation calculation can be made.

The polarization eigenvectors of the eigenmodes to the dispersion equations are then

$$\lambda^1_\mu = \quad (\mathcal{S}-\mathcal{P})(\mathcal{T}-\mathcal{P})a^{(1)}_\mu \tag{1.3a}$$

$$\lambda^2_\mu = -(\mathcal{P}-\mathcal{S})(\mathcal{T}-\mathcal{S})a^{(2)}_\mu \tag{1.3b}$$

$$\lambda^3_\mu = \quad (\mathcal{P}-\mathcal{T})(\mathcal{S}-\mathcal{T})a^{(3)}_\mu \ . \tag{1.3c}$$

The notation for the modes 1, 2, 3 concurs with that adopted by [SH75]. All the eigenmodes are linearly polarized modes with mode 3 being fully transversal. If $q_\perp \neq 0$ one can choose a Lorentz frame so that $q_{\|} = 0$ and then the electric field 3-vector of mode 1 is longitudinal, $e^{(1)} \| q$, while that of mode 2 is transverse. If the converse is true then the polarization vector of mode 1 becomes transverse, being perpendicular to the plane defined by the wavevector and the external field, while that of mode 2 becomes longitudinal. The unnormalized electric and magnetic polarization 3-vectors corresponding to the $a^{(1)}$, $a^{(2)}$ and $a^{(3)}$ are:

$$e^{(1)} = \Omega(|q|^2 - \Omega^2) q_\perp \qquad h^{(1)} = q_{\|}(|q|^2 - \Omega^2) q_{\|} \times q_\perp \tag{1.4a}$$

$$e^{(2)} = q_{\|}^2 q_\perp + (q_{\|}^2 - \Omega^2) q_{\|} \qquad h^{(2)} = \Omega q_{\|} \times q_\perp \tag{1.4b}$$

$$e^{(3)} = \Omega q_{\|} \times q_\perp \qquad h^{(3)} = q_\perp^2 q_{\|} - q_{\|}^2 q_\perp \,. \tag{1.4c}$$

However it is sometimes convenient to use the scalars originally defined by Bakshi *et al* and which are related to the other set by

$$\begin{aligned} \mathcal{P} &= (y+z)P \\ \mathcal{S} &= yQ + zP \\ \mathcal{T} &= yP + zR \end{aligned} \tag{1.5}$$

In this way the spatial part of the vacuum polarization tensor is

$$\Pi_{ij} = \begin{pmatrix} yP & 0 & -q_\perp q_{\|} P \\ 0 & yP + zR & 0 \\ -q_\perp q_{\|} P & 0 & zP - \Omega^2 Q \end{pmatrix} . \tag{1.6}$$

With this set of scalars the dispersion relation decouples into the following set of three simple equations:

$$1 + P = 0 \qquad \text{Mode 1} \tag{1.7a}$$

$$y(1+Q) + z(1+P) = 0 \qquad \text{Mode 2} \tag{1.7b}$$

$$y(1+P) + z(1+R) = 0 \qquad \text{Mode 3} \tag{1.7c}$$

and the most general solution can only relate y to z via the relation $y = y^M(z)$ for modes $M = 1, 2, 3$.

In Shabad [SH75] one has the first detailed description of the analyticity of the fermion vacuum scalars, calculated to the one-loop level, as functions in the complex y and z planes. The scalars are entire functions of the complex variable z. However their analyticity with respect to y is quite different, and each scalar has branch points on the negative real y axis corresponding to pair creation thresholds (which particular thresholds they are depends on the scalar). The branch cut runs from the highest threshold point to negative infinity. Thus, at this level of approximation for the fermion problem it was found in this work that the scalar functions possess two Riemann sheets in the cut complex y plane which arise out of the characteristic inverse square-root dependence on y of the singular factors in the scalars. This feature is explicitly demonstrated in appendix C of [WT89].

2. Proper-time representation

The spatial components of the polarization tensor are given by equations (5.23) in [WT88], which are:

$$\Pi_{11} = \frac{\alpha}{4\pi}\beta^2 \frac{e^{-z}}{2z} \sum_{n,n'=0}^{\infty} (n'-n)^2 \left[\Theta^0_{n',n}\right]^2 M^0_{n,n'} + \frac{\alpha}{2\pi}\beta N_{\mathrm{T}} \tag{2.1a}$$

$$\Pi_{12} = -i\frac{\alpha}{4\pi}\beta^2 e^{-z} {\mu_\perp}^{-1} \sum_{n,n'=0}^{\infty} (n'-n)\Theta^-_{n',n}\Theta^0_{n',n} M^0_{n,n'} \tag{2.1b}$$

$$\Pi_{13} = -\frac{\alpha}{4\pi}\beta^2 e^{-z} {\mu_\perp}^{-1} \sum_{n,n'=0}^{\infty} (n'-n)\left[\Theta^0_{n',n}\right]^2 b^{-1} M^1_{n,n'} \tag{2.1c}$$

$$\Pi_{22} = \frac{\alpha}{4\pi}\beta^2 e^{-z} \sum_{n,n'=0}^{\infty} \left[\Theta^-_{n',n}\right]^2 M^0_{n,n'} + \frac{\alpha}{2\pi}\beta N_{\mathrm{T}} \tag{2.1d}$$

$$\Pi_{23} = -i\frac{\alpha}{4\pi}\beta^2 e^{-z} \sum_{n,n'=0}^{\infty} \Theta^0_{n',n}\Theta^-_{n',n} b^{-1} M^1_{n,n'} \tag{2.1e}$$

$$\Pi_{33} = \frac{\alpha}{4\pi}\beta^2 e^{-z} \sum_{n,n'=0}^{\infty} \left[\Theta^0_{n',n}\right]^2 \left\{ b^{-2}\Omega J^2_{n,n'} - (n'-n)^2 {\mu_\parallel}^{-2} I^0_{n,n'} \right\} + \frac{\alpha}{2\pi}\beta {\mu_\perp}^2 {\mu_\parallel}^{-2} N_{\mathrm{T}}. \tag{2.1f}$$

N_{T} denotes the formally divergent sum

$$\sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \frac{d\mathcal{P}_\parallel}{\mathcal{E}(\mathcal{P}_\parallel, n)}.$$

The first step in calculating the renormalized boson vacuum tensor is the isolation of the '1' or vacuum contributions to the momentum integrals, that is to say the integrals $M^k_{n,n'}, I^k_{n,n'}$, and $J^k_{n,n'}$ defined by equations (5.22) in [WT88] as they appear in the polarization tensor components above. The vacuum parts take the following form

$$\begin{aligned} M^k_{n,n'}(q_\parallel, \Omega) = \int_{-\infty}^{+\infty} d\mathcal{P}_\parallel & \frac{{\mathcal{P}_\parallel}^k}{\mathcal{E}(\mathcal{P}_\parallel + \frac{1}{2}q_\parallel, n)\mathcal{E}(\mathcal{P}_\parallel - \frac{1}{2}q_\parallel, n')} \\ & \times \left\{ \left[\Omega - \mathcal{E}(\mathcal{P}_\parallel + \tfrac{1}{2}q_\parallel, n) - \mathcal{E}(\mathcal{P}_\parallel - \tfrac{1}{2}q_\parallel, n') + i\eta\right]^{-1} \right. \\ & \left. - \left[\Omega + \mathcal{E}(\mathcal{P}_\parallel + \tfrac{1}{2}q_\parallel, n) + \mathcal{E}(\mathcal{P}_\parallel - \tfrac{1}{2}q_\parallel, n') + i\eta\right]^{-1} \right\} \end{aligned} \tag{2.2a}$$

$$\begin{aligned} J^k_{n,n'}(q_\parallel, \Omega) = \int_{-\infty}^{+\infty} d\mathcal{P}_\parallel & \frac{{\mathcal{P}_\parallel}^k}{\mathcal{E}(\mathcal{P}_\parallel + \frac{1}{2}q_\parallel, n)\mathcal{E}(\mathcal{P}_\parallel - \frac{1}{2}q_\parallel, n')[\mathcal{E}(\mathcal{P}_\parallel + \frac{1}{2}q_\parallel, n) + \mathcal{E}(\mathcal{P}_\parallel - \frac{1}{2}q_\parallel, n')]} \\ & \times \left\{ \left[\Omega - \mathcal{E}(\mathcal{P}_\parallel + \tfrac{1}{2}q_\parallel, n) - \mathcal{E}(\mathcal{P}_\parallel - \tfrac{1}{2}q_\parallel, n') + i\eta\right]^{-1} \right. \\ & \left. + \left[\Omega + \mathcal{E}(\mathcal{P}_\parallel + \tfrac{1}{2}q_\parallel, n) + \mathcal{E}(\mathcal{P}_\parallel - \tfrac{1}{2}q_\parallel, n') + i\eta\right]^{-1} \right\} \end{aligned} \tag{2.2b}$$

$$I^k_{n,n'}(q_\parallel) = 2\int_{-\infty}^{+\infty} d\mathcal{P}_\parallel \frac{{\mathcal{P}_\parallel}^k}{\mathcal{E}(\mathcal{P}_\parallel + \frac{1}{2}q_\parallel, n)\mathcal{E}(\mathcal{P}_\parallel - \frac{1}{2}q_\parallel, n')[\mathcal{E}(\mathcal{P}_\parallel + \frac{1}{2}q_\parallel, n) + \mathcal{E}(\mathcal{P}_\parallel - \frac{1}{2}q_\parallel, n')]}. \tag{2.2c}$$

Here $\mathcal{E}(\mathcal{P}_{\|}, n) \equiv \left[1 + \mathcal{P}_{\|}{}^2 + \beta(n+\frac{1}{2})\right]^{1/2}$ and later use is made of the relation $\mathcal{E}_n \equiv \left[1 + \beta(n+\frac{1}{2})\right]^{1/2}$. These integrals are symmetric or antisymmetric with respect to exchange in the Landau level indices n and n' according to whether k is even or odd respectively. They are related to each other by $M^k_{n,n'} = \Omega J^k_{n,n'} - I^k_{n,n'}$. They are finite for $k = 0, 1$ in the case of M and I, and for $k = 0, 1, 2$ in the case of J.

The definition of the $\Theta^0_{n',n}$ and $\Theta^-_{n',n}$ functions are:

$$\Theta^0_{n',n}(z) = \alpha_{n',n}(\mu_\perp) \tag{2.3a}$$

$$\Theta^-_{n',n}(z) = \sqrt{\frac{n+1}{2}}\alpha_{n',n+1}(\mu_\perp) - \sqrt{\frac{n}{2}}\alpha_{n',n-1}(\mu_\perp) \tag{2.3b}$$

where

$$\alpha_{n',n}(\mu_\perp) = \begin{cases} \left[\frac{n!}{n'!} z^{n'-n}\right]^{1/2} L_n^{n'-n}(z) & \text{if } n' > n \\ (-1)^{n-n'} \left[\frac{n'!}{n!} z^{n-n'}\right]^{1/2} L_{n'}^{n-n'}(z) & \text{if } n > n' \end{cases} \tag{2.3c}$$

with $z \equiv \frac{1}{2}\mu_\perp^2$ (distinct from $z = \beta z$) and the $L_n^{n'-n}$ are the standard associated Laguerre polynomials as given in [GD80]. Appendix A of [WT88] has a list of properties of the Θ functions. From the symmetry of the M integral and the Θ functions under exchange of n and n' is follows that

$$\sum_{n,n'=0}^{\infty} \Theta^0_{n',n}\Theta^-_{n',n}(n'-n)M^0_{n,n'} = 0 \tag{2.4a}$$

$$\sum_{n,n'=0}^{\infty} \Theta^0_{n',n}\Theta^-_{n',n}M^1_{n,n'} = 0 \tag{2.4b}$$

so that $\Pi_{12} = \Pi_{23} = 0$. Thus this vacuum tensor conforms to the invariance principles outlined above. The momentum integrals required for the non-zero elements are those which are finite, but the double infinite sums over the Landau level numbers n and n' are formally divergent.

All the momentum integrals can be evaluated explicitly directly from their definitions but it is more useful to cast them in a different form which allows one to easily switch from one representation to another. Turning to the simplest example of $I^k_{n,n'}$ it is shown in [WT89] that

$$I^0_{n,n'} = 2\int_{-1/2}^{+1/2} \mathrm{d}x \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t\mathcal{Q}'_\mathrm{b}} \tag{2.5a}$$

$$I^1_{n,n'} = 2q_{\|}\int_{-1/2}^{+1/2} \mathrm{d}x\, x \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t\mathcal{Q}'_\mathrm{b}} \tag{2.5b}$$

where

$$\mathcal{Q}'_\mathrm{b}(t,x) \equiv 1 + q_{\|}^2(\tfrac{1}{4} - x^2) + \tfrac{1}{2}\beta(n+n'+1) + \beta(n'-n)x\ .$$

Considering the real parts of M and J next, one should note the presence of Ω in the denominator and it might seem that an equivalent result would not exist. However such a result does exist, despite appearances, as it is clear that the integral can only be a function of $q_{\|}$ and Ω combined in the relativistic scalar y and not separately. As described in [WT89] one finds the following results:

$$Re(M^0_{n,n'})(y) = -2\int_{-1/2}^{+1/2} \mathrm{d}x \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t\mathcal{Q}_\mathrm{b}} = -2\int_{-1/2}^{+1/2} \mathrm{d}x\, \mathcal{Q}_\mathrm{b}^{-1} \tag{2.6a}$$

$$Re(M^1_{n,n'})(y) = -2q_{\|}\int_{-1/2}^{+1/2} \mathrm{d}x\, x \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t\mathcal{Q}_\mathrm{b}} = -2q_{\|}\int_{-1/2}^{+1/2} \mathrm{d}x\, x\mathcal{Q}_\mathrm{b}^{-1} \tag{2.6b}$$

$$\begin{aligned}\Omega Re(J^2_{n,n'})(y) &= \int_{-1/2}^{+1/2} \mathrm{d}x \int_0^\infty \mathrm{d}t\, t^{-1}\left(\mathrm{e}^{-t\mathcal{Q}'_\mathrm{b}} - \mathrm{e}^{-t\mathcal{Q}_\mathrm{b}}\right)\\ &\quad + 2q_{\|}^2\int_{-1/2}^{+1/2} \mathrm{d}x\, x^2 \int_0^\infty \mathrm{d}t\, \left(\mathrm{e}^{-t\mathcal{Q}'_\mathrm{b}} - \mathrm{e}^{-t\mathcal{Q}_\mathrm{b}}\right)\\ &= \int_{-1/2}^{+1/2} \mathrm{d}x\, \ln\left[\mathcal{Q}_\mathrm{b}/\mathcal{Q}'_\mathrm{b}\right] + 2q_{\|}^2\int_{-1/2}^{+1/2} \mathrm{d}x\, x^2\left(\mathcal{Q}'^{-1}_\mathrm{b} - \mathcal{Q}_\mathrm{b}^{-1}\right)\end{aligned} \tag{2.6c}$$

where

$$\begin{aligned}\mathcal{Q}_\mathrm{b} &\equiv 1 + y(\tfrac{1}{4}-x^2) + \tfrac{1}{2}\beta(n+n'+1) + \beta(n'-n)x\ .\\ &= y(\tfrac{1}{4}-x^2) + \tfrac{1}{2}(\mathcal{E}^2_{n'}+\mathcal{E}^2_n) + x(\mathcal{E}^2_{n'}-\mathcal{E}^2_n).\end{aligned} \tag{2.7}$$

It is these forms for the integrals that will be of greatest use.

The x integrals in equations (2.6) are evaluated in appendix C of [WT89] and these illustrate explicitly the nature of the singularities at the pair production thresholds. As first described in [SH75] for the fermion integrals, the integrals of both cases are continuous across $y = -\{\mathcal{E}_{n'} - \mathcal{E}_n\}^2$, where $\mathcal{D}_\mathrm{b} = 0$, because $y + \mathcal{E}^2_{n'} + \mathcal{E}^2_n \to +2\mathcal{E}_{n'}\mathcal{E}_n$, but diverge with an inverse square-root singularity on the upper side of the $y = -\{\mathcal{E}_{n'} + \mathcal{E}_n\}^2$ pair threshold, where

$$\begin{aligned}\mathcal{D}_\mathrm{b}(n,n') &\equiv y^2 + 4y + 2y\beta(n+n'+1) + \beta^2(n-n')^2\\ &= \left[y + \{\mathcal{E}_n + \mathcal{E}_{n'}\}^2\right]\left[y + \{\mathcal{E}_n - \mathcal{E}_{n'}\}^2\right]\ .\end{aligned} \tag{2.8}$$

Here the difference is that $y + \mathcal{E}^2_{n'} + \mathcal{E}^2_n \to -2\mathcal{E}_{n'}\mathcal{E}_n$ even though $\mathcal{D}_\mathrm{b} = 0$ as before. On the other side of this singularity all the integrals have a finite limit because $|y + \mathcal{E}^2_{n'} + \mathcal{E}^2_n - \mathcal{D}_\mathrm{b}{}^{1/2}| \to +2\mathcal{E}_{n'}\mathcal{E}_n$. This particular and characteristic form was pointed out in the above reference to be due to the semi-discrete and continuously degenerate eigenspectrum of the free magnetic eigenstates, which form the basis of the one-loop approximations.

With the integrals in the above form of equations (2.5), (2.6) the double infinite Landau sum over n and n' appearing in equations (2.1) can be done exactly. The real parts of the integrals have a dependence on n and n' in the form of a factor

$$\exp\{-s\left[n(1/2-x) + n'(1/2+x)\right]\}$$

with $s \equiv \beta t$ and using the technique expounded in appendix B of [WT89] the two basic double sums required are:

$$\sum_{n,n'=0}^{\infty} \left[\Theta^0_{n',n}\right]^2 \exp\{-s[n(1/2-x)+n'(1/2+x)]\}$$
$$= (1-e^{-s})^{-1}\exp\left\{2z\left(\frac{e^{-s/2}\cosh(sx)-e^{-s}}{1-e^{-s}}\right)\right\} \qquad (2.8a)$$

$$\sum_{n,n'=0}^{\infty} \left[\Theta^-_{n',n}\right]^2 \exp\{-s[n(1/2-x)+n'(1/2+x)]\}$$
$$= \left[\frac{z}{2(1-e^{-s})^3}\left(2e^{-s/2}\cosh(sx)-1-e^{-s}\right)^2+\frac{e^{-s/2}}{(1-e^{-s})^2}\cosh(sx)\right]$$
$$\times \exp\left\{2z\left(\frac{e^{-s/2}\cosh(sx)-e^{-s}}{1-e^{-s}}\right)\right\} . \qquad (2.8b)$$

All other sums can be found from these two by differentiation with respect to x. An important single sum is

$$\sum_{n'=0}^{\infty} \left[\Theta^0_{n',n}\right]^2 = e^z . \qquad (2.8c)$$

The essence of this technique reduces to a recognition that the single charged-particle quantum problem in a uniform magnetic field is simply a product of two one-dimensional simple harmonic oscillator algebras and the sums are matrix elements in this algebra.

Assembling these results into the expressions for the polarization tensor, equations (2.1), rewriting the N_{T} terms in the form

$$N_{\mathrm{T}} = \int \mathrm{d}s\, s^{-1}(1-e^{-s})^{-1}e^{-s(1/2+\beta^{-1})}$$

so that they can be absorbed into the double sum, and integrating by parts with respect to x where necessary one arrives at the unregularized elements. The unregularized elements appear in precisely the correct form, given in equation (1.6), and one can make the identification of the three bare scalar invariants $Q_{\mathrm{b}}, P_{\mathrm{b}}$ and R_{b} as:

$$Q_{\mathrm{b}} = \frac{\alpha}{2\pi}\int \mathrm{d}x\, x^2 \int \mathrm{d}s\, \mathrm{cosech}(\tfrac{1}{2}s)e^{-t\varphi} \qquad (2.9a)$$

$$P_{\mathrm{b}} = \frac{\alpha}{4\pi}\int \mathrm{d}x\, x \int \mathrm{d}s\, \frac{\sinh(sx)}{\sinh^2(\frac{1}{2}s)}e^{-t\varphi} \qquad (2.9b)$$

$$Q_{\mathrm{b}} = -\frac{\alpha}{8\pi}\int \mathrm{d}x \int \mathrm{d}s\, \frac{1+\cosh^2(\frac{1}{2}s)-2\cosh(\frac{1}{2}s)\cosh(sx)}{\sinh^3(\frac{1}{2}s)}e^{-t\varphi} \qquad (2.9c)$$

where

$$\varphi = 1+y(\tfrac{1}{4}-x^2)+z\left\{\frac{\cosh(\frac{1}{2}s)-\cosh(sx)}{s\sinh(\frac{1}{2}s)}\right\} .$$

The renormalization procedure employed here involves the subtraction of the zero-field limit of the unregularized scalars from the non-zero field originals and the addition of the exact renormalized zero-field tensor. Applying this to an arbitrary scalar $S(\boldsymbol{q},\Omega|\beta)$ leads to

$$\overline{S}(\boldsymbol{q},\Omega|\beta) = S(\boldsymbol{q},\Omega|\beta) - \lim_{\beta\to 0} S(\boldsymbol{q},\Omega|\beta) + \overline{S}(\boldsymbol{q},\Omega|0) \tag{2.10}$$

where the regularized quantity is distinguished from the unregularized by the bar. This is the most direct subtraction process, but is only suited to those cases where the form of the unregularized finite field scalar is similar to the zero-field limit in order to simply effect the subtraction. When this is not the case intermediate steps are required in which the $\boldsymbol{q}$ components and Ω are set to zero one at a time in some order.

In the zero-field limit φ reduces to the isotropic form $\varphi_0 = 1 + (y + z)(\frac{1}{4} - x^2)$ and one arrives at

$$Q_\mathrm{b} - \lim_{\beta\to 0} Q_\mathrm{b} = \frac{\alpha}{2\pi}\int_{-1/2}^{+1/2} \mathrm{d}x\, x^2 \int_0^\infty \mathrm{d}s \left[\mathrm{cosech}(\tfrac{1}{2}s)\mathrm{e}^{-t\varphi} - \frac{2}{s}\mathrm{e}^{-t\varphi_0}\right] \tag{2.11a}$$

$$P_\mathrm{b} - \lim_{\beta\to 0} P_\mathrm{b} = \frac{\alpha}{4\pi}\int_{-1/2}^{+1/2} \mathrm{d}x\, x \int_0^\infty \mathrm{d}s \left[\frac{\sinh(sx)}{\sinh^2(\frac{1}{2}s)}\mathrm{e}^{-t\varphi} - \frac{4x}{s}\mathrm{e}^{-t\varphi_0}\right] \tag{2.11b}$$

$$R_\mathrm{b} - \lim_{\beta\to 0} R_\mathrm{b} = \frac{\alpha}{8\pi}\int_{-1/2}^{+1/2} \mathrm{d}x \int_0^\infty \mathrm{d}s \times \left[\frac{2\cosh(\frac{1}{2}s)\cosh(sx) - 1 - \cosh^2(\frac{1}{2}s)}{\sinh^3(\frac{1}{2}s)}\mathrm{e}^{-t\varphi} - \frac{8x^2}{s}\mathrm{e}^{-t\varphi_0}\right] . \tag{2.11c}$$

To complete the renormalization the regularized zero-field tensor is required and this has been calculated before [KW85] (their result should be corrected). This is characterized by one scalar and can be recast into a form appropriate to equations (2.11) as

$$Re(\Pi_0) = \frac{\alpha}{\pi}\int_{-1/2}^{+1/2} \mathrm{d}x\, x^2 \int_0^\infty \mathrm{d}s\, s^{-1} \left[\mathrm{e}^{-s\varphi_0} - \mathrm{e}^{-s}\right] . \tag{2.12}$$

Finally one finds some internal cancellation upon addition of this last term to equations (2.11) and the proper-time representation of the magnetized boson vacuum scalars are:

$$\overline{Q}_\mathrm{b} = \frac{\alpha}{2\pi}\int_{-1/2}^{+1/2} \mathrm{d}x\, x^2 \int_0^\infty \mathrm{d}s \left[\mathrm{cosech}(\tfrac{1}{2}s)\mathrm{e}^{-t\varphi} - \frac{2}{s}\mathrm{e}^{-t}\right] \tag{2.13a}$$

$$\overline{P}_\mathrm{b} = \frac{\alpha}{4\pi}\int_{-1/2}^{+1/2} \mathrm{d}x\, x \int_0^\infty \mathrm{d}s \left[\frac{\sinh(sx)}{\sinh^2(\frac{1}{2}s)}\mathrm{e}^{-t\varphi} - \frac{4x}{s}\mathrm{e}^{-t}\right] \tag{2.13b}$$

$$\overline{R}_\mathrm{b} = \frac{\alpha}{8\pi}\int_{-1/2}^{+1/2} \mathrm{d}x \int_0^\infty \mathrm{d}s \left[\frac{2\cosh(\frac{1}{2}s)\cosh(sx) - 1 - \cosh^2(\frac{1}{2}s)}{\sinh^3(\frac{1}{2}s)}\mathrm{e}^{-t\varphi} - \frac{8x^2}{s}\mathrm{e}^{-t}\right] . \tag{2.13c}$$

The above equations are valid representations for y values greater than the first pair-production threshold of the particular scalar. Taking the case of $\overline{Q}_{\rm b}$, this can be seen from the large s expansion of the first term in the integrand

$$\mathrm{e}^{-t\varphi}\,\mathrm{cosech}(\tfrac{1}{2}s) \to 2\mathrm{e}^{-s(g+1/2)} \tag{2.14a}$$

where $g \equiv \beta^{-1}[1 + y(\frac{1}{4} - x^2)]$ and to ensure that the argument $g + \frac{1}{2}$ is positive for all x then the minimum of this quantity must positive. This implies $y > -\{\mathcal{E}_0 + \mathcal{E}_0\}^2$. In the case of $\overline{P}_{\rm b}$ the limit is

$$\mathrm{e}^{-t\varphi}\sinh(sx)/\sinh^2(\tfrac{1}{2}s) \to 2\mathrm{sgn}(x)\mathrm{e}^{-s(g+1-|x|)} \tag{2.14b}$$

and the appropriate condition is $y > -\{\mathcal{E}_0 + \mathcal{E}_1\}^2$, while the case of $\overline{R}_{\rm b}$ has a limit

$$\mathrm{e}^{-t\varphi}\left(\mathrm{e}^{sx} - \cosh(\tfrac{1}{2}s)\right)\left(\cosh(\tfrac{1}{2}s) - \mathrm{e}^{-sx}\right)/\sinh^3(\tfrac{1}{2}s) \to -2\mathrm{e}^{-s(g+1/2)} \tag{2.14c}$$

and therefore $y > -\{\mathcal{E}_0 + \mathcal{E}_0\}^2$. Similar considerations for the fermions leads to $y > -4$ in the case of $\overline{Q}_{\rm f}$, to $y > -\{\mathcal{E}_0 + \mathcal{E}_1\}^2$ in the case of $\overline{P}_{\rm f}$, and and to $y > -\{\mathcal{E}_1 + \mathcal{E}_1\}^2$ in the case of $\overline{R}_{\rm f}$. In all these equations and the following ones the same symbol for the energy level of both fermions and bosons is used, namely $\mathcal{E}_n$, but it should be clear which is meant from the context.

The proper-time integral formulation can be analytically continued throughout the complex y plane except on a branch cut from the pair threshold to $-\infty$. This continuation entails a rotation of integration contour from $s = 0 \to +\infty$ to any path between $s = 0 \to +\mathrm{i}\infty$ and $s = 0 \to -\mathrm{i}\infty$. The complex integral form corresponding to a rotation of $+\pi/2$ (and a change of variables $s \to \mathrm{i}s$) is essentially identical in appearance to the real form and only the result for $\overline{Q}_{\rm b}$ will be quoted here:

$$\overline{Q}_{\rm b} = \frac{\alpha}{2\pi}\int_{-1/2}^{+1/2} \mathrm{d}x\, x^2 \int_0^{\infty} \mathrm{d}s \left[\mathrm{cosec}(\tfrac{1}{2}s)\mathrm{e}^{-\mathrm{i}t\psi} - \frac{2}{s}\mathrm{e}^{-\mathrm{i}t}\right] \tag{2.15}$$

where

$$\psi = 1 + y(\tfrac{1}{4} - x^2) + z\left\{\frac{\cos(\frac{1}{2}s) - \cos(sx)}{s\,\sin(\frac{1}{2}s)}\right\} .$$

The precise contour is indented along semicircles of arbitrarily small radii centred around the poles of the integrand at $s = s_n = 2\pi n$, where $n = 1, 2, 3, \ldots$, in the lower half $\mathrm{Im}\, s < 0$ plane (there is no pole at $s = 0$ due to the cancellation with the counter term). Effectively the contour runs from $s = 0$ to $s = \infty\mathrm{e}^{-\mathrm{i}\epsilon}$ for some small positive ϵ and this introduces a convergence factor $\mathrm{e}^{-t\epsilon\psi}$ to ensure the existence of the integral. However this form will find no particular application in the work presented here, and is unsuitable for numerical work. The final results for the boson scalars in equations (2.13a–c) can be compared to the special case of the polarization tensor in a pure external magnetic field that was found in [BI75], and exact agreement is found by making the correspondences $x \to \frac{1}{2}s$, $v \to 2x$, $\mu^{-1}x \to s/\beta$ and $2H/H_0 \to \beta$.

3. Spectral representation

In this section the spectral or Landau sum representation of the boson scalars will be derived. The main usefulness of such a representation is in the interpretation of the individual terms and the approximations that can be found from it.

Considering Q_b first, the following unrenormalized equation for Π_{33} is taken as the starting point:

$$\Pi_{33} = \frac{\alpha}{2\pi}\beta e^{-z} \sum_{n,n'=0}^{\infty} \left[\Theta^0_{n',n}\right]^2 M^2_{n,n'} + \frac{\alpha}{2\pi}\beta N_\mathrm{T} \tag{3.1a}$$

in contrast to that in equation (2.2f). Here the integrals $M^2_{n,n'}$ and N_n defined to be

$$\begin{aligned} M^2_{n,n'} &= -\int dx \int dt\, t^{-1} e^{-tQ_\mathrm{b}} - 2q_{\parallel}^2 \int dx\, x^2 \int dt\, e^{-tQ_\mathrm{b}} \\ N_n &= \int dx \int dt\, t^{-1} e^{-t\mathcal{E}_n^2} \\ N_\mathrm{T} &= \sum_{n=0}^{\infty} N_n \, . \end{aligned} \tag{3.1b}$$

The t integrals are formally divergent at the lower limit of $t = 0$, but one can take this limit as being small and finite in intermediate steps until it is set to zero at the end. How this step in the renormalization process works here is that the two divergent integrals in the component can be combined to yield a finite integral, by using the sum identity, equation (2.8c). The tensor component is found to be:

$$\begin{aligned} \Pi_{33} = {} & \frac{\alpha}{4\pi}\beta e^{-z} \sum_{n,n'=0} \left[\Theta^0_{n',n}\right]^2 \int_{-1/2}^{+1/2} dx \left\{ \ln\left[\frac{Q_\mathrm{b}(-x)}{Q_\mathrm{b}(+1/2)}\right] + \ln\left[\frac{Q_\mathrm{b}(x)}{Q_\mathrm{b}(-1/2)}\right] \right\} \\ & - \frac{\alpha}{\pi}\beta q_{\parallel}^2 e^{-z} \sum_{n,n'=0} \left[\Theta^0_{n',n}\right]^2 \int_{-1/2}^{+1/2} dx\, x^2 Q_\mathrm{b}^{-1}(x) \, . \end{aligned} \tag{3.2}$$

By integrating the first term by parts, one finds a single term with a factor of Ω^2. On the basis of the structure of the vacuum tensor one can make the identification of this term with Q_b. The unrenormalized scalar found this way is then

$$Q_\mathrm{b} = \frac{\alpha}{\pi}\beta e^{-z} \sum_{n,n'=0} \left[\Theta^0_{n',n}\right]^2 \int_{-1/2}^{+1/2} dx\, x^2 Q_\mathrm{b}^{-1}(x) \, . \tag{3.3}$$

The unregularized scalar P_b is most conveniently found from the Π_{13} component directly (equation (2.1c)) and is

$$P_\mathrm{b} = -\frac{\alpha}{2\pi}\beta e^{-z} \sum_{n,n'=0} \frac{(n'-n)}{z} \left[\Theta^0_{n',n}\right]^2 \int_{-1/2}^{+1/2} dx\, x Q_\mathrm{b}^{-1} \, . \tag{3.4}$$

The remaining boson scalar is found in a similar way with the unregularized form extracted from the Π_{22} component:

$$R_\mathrm{b} = \frac{\alpha}{2\pi}\beta \frac{e^{-z}}{z} \sum_{n,n'=0} \left\{ \frac{(n'-n)^2}{2z} \left[\Theta^0_{n',n}\right]^2 - \left[\Theta^-_{n',n}\right]^2 \right\} \int_{-1/2}^{+1/2} dx\, Q_\mathrm{b}^{-1} \, . \tag{3.5}$$

Renormalization in this representation proceeds in a slightly different way from that of the previous section, because the limit $\lim_{\beta\to 0}\Theta^0_{n',n}(z/\beta)$, for general z, n and n' is not known. This was a point noted by Sivak [SV85] but not resolved. However one does not need this, and instead proceeds to take the limit of $z \to 0$ first because in the zero-field limit quantities only depend on an isotropic $y \to q^2 - \Omega^2$, then $y \to 0$ and finally $\beta \to 0$ in distinct steps. Having performed the first two limits one performs the zero field limit by utilizing the technique employed by [BK75, BK76]. If $F(\beta n, \beta, x)$ is an arbitrary integrand of a x-integral and a function of the combination βn, β alone and x, then one can recast this limit of the infinite sum over n as an integral thus:

$$\lim_{\beta\to 0}\beta\sum_{n=0}^{\infty}\int_{-1/2}^{+1/2}\mathrm{d}x\, F(\beta n,\beta,x) = \int_0^{\infty}\mathrm{d}u\int_{-1/2}^{+1/2}\mathrm{d}x\, F(u,0,x)$$
$$= \beta\sum_{n=0}^{\infty}\int_0^1\mathrm{d}y\int_{-1/2}^{+1/2}\mathrm{d}x\, F(\beta(n+y),0,x). \tag{3.6}$$

The final limit required for Q_b is

$$\lim_{\beta\to 0}\lim_{y,z\to 0} = \frac{\alpha}{12\pi}\beta\sum_{n=0}\int_0^1\mathrm{d}y\,[1+\beta(n+y)]^{-1} \tag{3.7}$$

and the renormalized scalar is then

$$\overline{Q}_\mathrm{b} = \frac{\alpha}{\pi}\beta\sum_{n=0}^{\infty}\left\{\mathrm{e}^{-z}\sum_{n'=0}^{\infty}\left[\Theta^0_{n',n}\right]^2\int_{-1/2}^{+1/2}\mathrm{d}x\, x^2 {Q_\mathrm{b}}^{-1} - \tfrac{1}{12}\beta^{-1}\ln\left[\frac{1+\beta(n+1)}{1+\beta n}\right]\right\}. \tag{3.8}$$

It should be noted that the order of the summations is specific. Firstly the n' sum is done and is convergent for all n, y and z because for large n' the Laguerre polynomials go as $L_n^{n'-n}(z) \sim (1+z)^n/n!$ and thus the theta functions behave like

$$\left[\Theta^0_{n',n}\right]^2 \sim \frac{z^{n'-n}(1+z)^{2n}}{n'!n!}$$

and the integrals are decreasing functions of n'. For each n this partial sum is then renormalized by the subtraction term and the n summation becomes convergent, although weakly so. The subtraction term can be put into a form symmetrical in n and n' by noting that the index n is arbitrary and utilizing the sum identity (2.8c) once more. This yields the expression

$$\overline{Q}_\mathrm{b} = \frac{\alpha}{\pi}\beta\sum_{n,n'=0}^{\infty}\mathrm{e}^{-z}\left[\Theta^0_{n',n}\right]^2$$
$$\times\left\{\int_{-1/2}^{+1/2}\mathrm{d}x\, x^2 {Q_\mathrm{b}}^{-1} - \tfrac{1}{24}\beta^{-1}\ln\left[\frac{1+\beta(n+1)}{1+\beta n}\cdot\frac{1+\beta(n'+1)}{1+\beta n'}\right]\right\} \tag{3.9}$$

and the order of summations is irrelevant. For example one can sum over n along the diagonal $n'-n = \nu =$ constant lines, where $\nu = 1, 2, 3\ldots$, and then sum over ν, and this will yield the correct result.

The renormalized spectral form can also be developed directly from the renormalized proper-time representation, and to illustrate the method Q_f is taken as an example. The first calculation of these forms in the fermion case appeared in Shabad [SH75] where the unrenormalized scalars were displayed. The contact terms were not explicitly shown. These representations have been found also by Sivak [SV85], who gave the most compact expressions, but they differ from the ones presented here in that his subtraction terms are not the zero-field limits. They are in fact just the $y \to 0$ and $z \to 0$ limits and he has to add on the finite field static and uniform renormalized scalar. Melrose [MR77] has also found them, but not simplified or reduced them in the same way as the other two authors. In the derivation here the factor $e^{-t\varphi}/\sinh(\frac{1}{2}s)$ is expanded, using the double Landau sum equation (2.8a), i.e. proceeding in the reverse manner to that detailed in the previous section. The contact term is expressed as

$$\frac{1}{s} = \int_{-1/2}^{+1/2} dx\, e^{-s(1/2+x)}(1-e^{-s})^{-1}$$

and expanded. The divergence of the s-integral as $s_L \to 0$, where s_L is the lower limit of the s-integral, is now transfered to the double infinite sum, and the s-integral performed. The final result is:

$$\overline{Q}_f = \frac{\alpha}{4\pi}\beta \sum_{n=0}^{\infty} \left\{ e^{-z} \sum_{n'=0}^{\infty} [\Theta^0_{n',n}]^2 \int_{-1/2}^{+1/2} dx\,(1-4x^2)\left[\mathcal{Q}_f^{-1}(n,n') + \mathcal{Q}_f^{-1}(n+1,n'+1)\right] - \tfrac{4}{3}\beta^{-1}\ln\left[\frac{1+\beta(n+1)}{1+\beta n}\right]\right\} . \tag{3.10}$$

This contact term can also be symmetrized and the sum identity (2.8c) used to perform a renormalization subtraction of each n, n' integral term as in equation (3.9). Here $\mathcal{Q}_f$ is the fermion equivalent of equation (2.7).

To implement the zero-field limiting procedure for the scalar P_b it is necessary to integrate the unrenormalized expression, equation (3.4), by parts with respect to x and then it is found that the contact term is

$$\lim_{\beta\to 0}\lim_{y,z\to 0} P_b = \frac{\alpha}{12\pi}\beta \sum_{n=0} \int_0^1 dy\, \beta(n+y)\,[1+\beta(n+y)]^{-2} . \tag{3.11}$$

Therefore the renormalized expression for this scalar is

$$\overline{P}_b = -\frac{\alpha}{2\pi}\beta \sum_{n=0}^{\infty} \left\{ e^{-z} \sum_{n'=0}^{\infty} \frac{(n'-n)}{z} [\Theta^0_{n',n}]^2 \int_{-1/2}^{+1/2} dx\, x\mathcal{Q}_b^{-1}(n,n') + \tfrac{1}{6}\beta^{-1}\left(\ln\left[\frac{1+\beta(n+1)}{1+\beta n}\right] + [1+\beta(n+1)]^{-1} - [1+\beta n]^{-1}\right)\right\} . \tag{3.12}$$

The contact term for R_b can be found from twice integrating by parts the bare term, equation (3.5), and the result of the limiting procedures is:

$$\lim_{\beta\to 0}\lim_{y,z\to 0} R_b = \frac{\alpha}{4\pi}\beta \sum_{n=0} \int_0^1 dy\, \left\{1+\tfrac{1}{3}\beta^2(n+y)^2\right\}[1+\beta(n+y)]^{-3} . \tag{3.13}$$

The final result for the renormalized scalar is then

$$\overline{R}_{\rm b} = \frac{\alpha}{2\pi}\beta\sum_{n=0}^{\infty}\left\{\frac{{\rm e}^{-z}}{z}\sum_{n'=0}^{\infty}\left[\frac{(n'-n)^2}{2z}\left[\Theta^0_{n',n}\right]^2 - \left[\Theta^-_{n',n}\right]^2\right]\int_{-1/2}^{+1/2}{\rm d}x\, Q_{\rm b}^{-1}(n,n') \right. \quad (3.14)$$
$$\left. -\tfrac{1}{6}\beta^{-1}\ln\left[\frac{1+\beta(n+1)}{1+\beta n}\right] - \frac{1}{3}\frac{n+1}{[1+\beta(n+1)]^2} + \frac{1}{3}\frac{n}{[1+\beta n]^2}\right\} .$$

The simple symmetrization scheme that was applied to the Q scalars is not a valid operation for the P or R scalars (either boson or fermion), and will lead to a finite but incorrect result. This is because the order of the infinite summations is important and cannot be interchanged without introducing modified contact terms and is ultimately due to the fact that they are not uniformly convergent sums in their own right.

4. Dispersion-sum representation

It is possible to develop another representation of the polarization tensor which is valid throughout the complex y plane apart from, obviously, the pair threshold branch points. The beginnings of a derivation of this representation was sketched out in the appendix of Shabad's 1975 paper [SH75] but no explicit final results were displayed. In this section the derivation of the result for $Q_{\rm b}$ will be given as an example, and this method can serve as a guide for the treatment of the other scalars.

Following [SH75] one starts with the proper-time representation, equation (2.13a), and expands ${\rm e}^{-t\varphi}$ as a power series in z about $z = 0$:

$${\rm e}^{-t\varphi} = \sum_{n=0}^{\infty}\frac{(-\frac{1}{2}z)^n}{n!}\left[\sinh(\tfrac{1}{2}s)\right]^{-n}\sum_{p,q=0}^{n}(-1)^{(p+q)}\,{}^nC_p\,{}^nC_q\,{\rm e}^{-sQ} \quad (4.1)$$

where $Q = g-\frac{1}{2}(p+q)+x(q-p)+\frac{1}{2}n$. One separates the $n = 0$ term and combines it with the contact term, i.e. the $n = 0$ term is renormalized, and the other $n \geq 1$ are not. After performing the s integration with care the expression for $\overline{Q}_{\rm b}$ is then

$$\frac{2\pi}{\alpha}Q_{\rm b} = -\tfrac{1}{6}\ln\beta - 2\int_{-1/2}^{+1/2}{\rm d}x\, x^2\psi(g+\tfrac{1}{2})$$
$$-2\sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{p,q=0}^{n}(-1)^{(p+q)}\,{}^nC_p\,{}^nC_q\int_{-1/2}^{+1/2}{\rm d}x\, x^2\frac{\Gamma(h_{\rm b}+n)}{n!\Gamma(h_{\rm b})}\psi(h_{\rm b}+n). \quad (4.2a)$$

Here $\psi(x)$ is the standard gamma-psi function and $h_{\rm b} \equiv g - \frac{1}{2}(p+q) + x(q-p) + \frac{1}{2}$. In this way the fermion scalar $\overline{Q}_{\rm f}$ is

$$\frac{\pi}{\alpha}Q_{\rm f} = -\tfrac{1}{3}\ln\beta - \int_{-1/2}^{+1/2}{\rm d}x\,(\tfrac{1}{4}-x^2)\left[\psi(g)+\psi(g+1)\right] - \sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{p,q=0}^{n}(-1)^{(p+q)}\,{}^nC_p\,{}^nC_q$$
$$\times\int_{-1/2}^{+1/2}{\rm d}x\,(\tfrac{1}{4}-x^2)\left\{\frac{\Gamma(h_{\rm f}+n)}{n!\Gamma(h_{\rm f})}\psi(h_{\rm f}) + \frac{\Gamma(h_{\rm f}+n+1)}{n!\Gamma(h_{\rm f}+1)}\psi(h_{\rm f}+n+1)\right\} \quad (4.2b)$$

where $h_{\rm f} \equiv g-\frac{1}{2}(p+q)+x(q-p)$. A derivation of and full results for all the other boson and fermion scalars, in this representation, can be found in [WT89]. The above formulae are convergent expansions in a double sense: in one sense as an expansion in the perpendicular wavevector z, and also as a high field expansion, in β^{-1}.

5. Anti-Hermitian parts

The anti-Hermitian parts of the vacuum tensor are exactly those corresponding to the imaginary parts of the scalars, as calculated from the components in equations (2.2) by the Landau procedure. From the optical theorem these are directly related to the inverse lifetimes of the modes supported by the magnetized vacuum. The inverse lifetime $\Re^M$ for an excitation mode M is not a Lorentz invariant but the product of the lifetime and frequency is. This product, calculated on the mass shell for the particular mode, is directly related to the anti-Hermitian part of the vacuum tensor, denoted by $\overline{\Pi}$, via the approximate relation

$$\Omega^M \Re^M = -\lambda^{M\dagger} \cdot \overline{\Pi} \cdot \lambda^M \; . \tag{5.1}$$

In the case of the three modes the Lorentz invariant rates are simply expressable in terms of the scalars:

$$(1) \qquad \Omega\Re^1 = -(y+z)\mathrm{Im}(\mathrm{P}) \tag{5.2a}$$

$$(2) \qquad \Omega\Re^2 = -y\mathrm{Im}(\mathrm{Q}) - z\mathrm{Im}(\mathrm{P}) \tag{5.2b}$$

$$(3) \qquad \Omega\Re^3 = -y\mathrm{Im}(\mathrm{P}) - z\mathrm{Im}(\mathrm{R}) \; . \tag{5.2c}$$

The limitations of this approximation will be discussed at greater length in section 8, which treats the exact eigensolutions of the dispersion relations.

The condition that a given term of the Landau sum, (n, n'), contributes a non-zero value to the anti-Hermitian part leads to a restriction on the permissible values of n and n' (if any are possible at all) for a given y and β, with the following form:

$$1 > \sqrt{l+l_0} + \sqrt{l'+l_0} \tag{5.3a}$$

$$1 < \left|\sqrt{l+l_0} - \sqrt{l'+l_0}\right| \tag{5.3b}$$

where $l \equiv -\beta n/y$, $l' \equiv -\beta n'/y$ and $l_0 \equiv -(1+\frac{1}{2}\beta)/y$. Shown in figure 1 are five distinct types of curves in the l against l' plane of the boundaries of these inequalities, which include the cases of $l_0 < 0$, $0 < l_0 < \frac{1}{4}$ and $l_0 > \frac{1}{4}$. The first case corresponds to $y > 0$ and there is no intersection with the axes. In the second case $y < 0$ and the boundary curve cuts the axes twice. Only in this case is there a region confined to the lower left-hand corner of the first quadrant of the l–l' plane which satisfies the pair production criteria of equation (5.3*a*). This criteria bounds the range of permissible Landau levels from above to only those that are kinematically possible in a pair creation process from a photon with 'energy' y. In the last case the boundary curve intersects the axes once and there are only regions satisfying the second equation (5.3*b*). The second inequality is the kinematic condition for gyromagnetic emission or absorption and is not present in the vacuum.

Because the sum over Landau levels in the imaginary parts has a finite cut-off and the zero-field and wavevector-frequency limits all vanish, they do not need to be regularized. The final result for the imaginary parts of the scalars is:

$$\mathrm{Im}(\overline{Q}_\mathrm{b}) = \tfrac{1}{2}\alpha\beta y^{-2}\mathrm{e}^{-z} \sum_{n,n'=0} \theta(-y_{n,n'}-y)\left[\Theta^0_{n',n}\right]^2 \times \left\{\mathcal{D}_\mathrm{b}{}^{1/2}(n,n') + \beta^2(n-n')^2\mathcal{D}_\mathrm{b}{}^{-1/2}(n,n')\right\} \tag{5.4a}$$

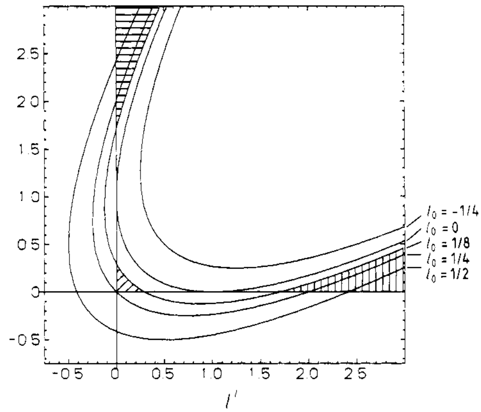

Figure 1. Plot of the regions in the Landau level plane l against l' that satisfy the pair creation or gyromagnetic absorption energy conditions for given values of field strength, β, and photon 'energy' parameter $\mathcal{Y}$.

$$\mathrm{Im}(\overline{P}_{\mathrm{b}}) = -\tfrac{1}{2}\alpha\beta(\mathcal{Y}z)^{-1}\mathrm{e}^{-z} \sum_{n,n'=0} \theta(-\mathcal{Y}_{n,n'}-\mathcal{Y}) \left[\Theta^{0}_{n',n}\right]^2 \beta^2(n-n')^2\mathcal{D}_{\mathrm{b}}{}^{-1/2}(n,n') \tag{5.4b}$$

$$\mathrm{Im}(\overline{R}_{\mathrm{b}}) = -\alpha\beta^2 z^{-1}\mathrm{e}^{-z} \sum_{n,n'=0} \theta(-\mathcal{Y}_{n,n'}-\mathcal{Y}) \times \left\{ \left[\Theta^{-}_{n',n}\right]^2 - \frac{1}{2z}(n-n')^2 \left[\Theta^{0}_{n',n}\right]^2 \right\} \mathcal{D}_{\mathrm{b}}{}^{-1/2}(n,n') \tag{5.4c}$$

where it is defined that $\mathcal{Y}_{n,n'} \equiv \{\mathcal{E}_{n'} + \mathcal{E}_n\}^2$.

The boson inverse lifetimes are found easily from equations (5.4) and are:

$$\Omega\Re^2_{\mathrm{b}} = -\tfrac{1}{2}\alpha\beta\mathcal{Y}^{-1}\mathrm{e}^{-z} \sum_{n',n=0} \left[\Theta^{0}_{n',n}\right]^2 \theta(-\mathcal{Y}_{n,n'}-\mathcal{Y})\mathcal{D}_{\mathrm{b}}{}^{+1/2}(n,n') \tag{5.5a}$$

and

$$\Omega\Re^3_{\mathrm{b}} = \alpha\beta\mathrm{e}^{-z} \sum_{n,n'=0} \left[\Theta^{-}_{n',n}\right]^2 \theta(-\mathcal{Y}_{n,n'}-\mathcal{Y})\mathcal{D}_{\mathrm{b}}{}^{-1/2}(n,n') \ . \tag{5.5b}$$

The inverse lifetime for the boson mode 2, $\Re^2_{\mathrm{b}}$, has no singularities anywhere, for all $\mathcal{Y}$ and z, in contrast to any of the two fermion inverse lifetimes. This arises from the constraints of angular momentum conservation of a process in which a spin-1 photon mode decays via pair creation of a 0-spin boson and antiboson pair. However, in a more exact calculation of the inverse lifetime from the dispersion relations all the fermion and boson lifetimes will have a finite maxima near the thresholds, and will not diverge (see discussion in [SH75, SH84]).

6. Static and uniform limits

In the investigation of the physical properties of the magnetized vacuum, it is instructive to begin with the simplest limit, the static and uniform response of the vacuum to a constant and homogeneous perturbing electromagnetic field. This limit can be most easily found from the dispersion sum representation, as given in equations (4.2*a*, *b*) etc, and the final results are displayed as follows:

$$Q_{\rm b}|_{y=z=0} = -\frac{\alpha}{12\pi}\left\{\ln\beta + \psi(\tfrac{1}{2}+\beta^{-1})\right\} \tag{6.1a}$$

$$P_{\rm b}|_{y=z=0} = \frac{\alpha}{12\pi} - \frac{\alpha}{12\pi}\ln\beta - \frac{\alpha}{\pi}\int_{-1/2}^{+1/2} \mathrm{d}x\,(x+\beta^{-1})x\psi(1+x+\beta^{-1}) \tag{6.1b}$$

$$R_{\rm b}|_{y=z=0} = \frac{\alpha}{2\pi}\beta^{-1} - \frac{\alpha}{12\pi}\ln\beta + \frac{\alpha}{\pi}\beta^{-2}\psi(\tfrac{1}{2}+\beta^{-1}) - \frac{\alpha}{\pi}\int_{-1/2}^{+1/2}\mathrm{d}x\,(x+\beta^{-1})^2\psi(1+x+\beta^{-1}). \tag{6.1c}$$

Landau sum representations for the static and uniform boson scalars can be found and these can be compared with the fermion equivalents in [BK75]. They can be found from the $z, y \to 0$ limits of expressions (3.8), (3.12) and (3.14) and the results displayed below can be shown to be exactly equal to those given in equations (6.1*a*–*c*). These are found to be

$$Q_{\rm b}|_{y=z=0} = \frac{\alpha}{12\pi}\sum_{n=0}^{\infty}\left\{[n+\tfrac{1}{2}+\beta^{-1}]^{-1} - \ln\left[\frac{n+1+\beta^{-1}}{n+\beta^{-1}}\right]\right\} \tag{6.2a}$$

and

$$\begin{aligned} P_{\rm b}|_{y=z=0} = \frac{\alpha}{2\pi}\sum_{n=0}^{\infty}\Bigg\{ & -2n-1 + 2(n+1)[n+1+\beta^{-1}]\ln\left[\frac{\mathcal{E}_{n+1}}{\mathcal{E}_n}\right] \\ & - 2n[n+\beta^{-1}]\ln\left[\frac{\mathcal{E}_{n-1}}{\mathcal{E}_n}\right] - \tfrac{1}{6}\ln\left[\frac{n+1+\beta^{-1}}{n+\beta^{-1}}\right] \\ & - \tfrac{1}{6}[1+\beta(n+1)]^{-1} + \tfrac{1}{6}[1+\beta n]^{-1}\Bigg\} \end{aligned} \tag{6.2b}$$

as well as

$$\begin{aligned} R_{\rm f}|_{y=z=0} = \frac{\alpha}{2\pi}\sum_{n=0}^{\infty}\Bigg\{ & 2(n+1)^2\ln\left[\frac{\mathcal{E}_{n+1}}{\mathcal{E}_n}\right] + 2n^2\ln\left[\frac{\mathcal{E}_n}{\mathcal{E}_{n-1}}\right] - \frac{1}{2}\frac{(2n+1)^2}{[n+\frac{1}{2}+\beta^{-1}]} \\ & - \tfrac{1}{6}\ln\left[\frac{n+1+\beta^{-1}}{n+\beta^{-1}}\right] - \tfrac{1}{3}\beta^{-1}\frac{(n+1)}{[n+1+\beta^{-1}]^2} + \tfrac{1}{3}\beta^{-1}\frac{n}{[n+\beta^{-1}]^2}\Bigg\}\,. \end{aligned} \tag{6.2c}$$

Simple approximations for the static and uniform limits exist in the low-field and high-field regions which expresses their behaviour in these regions in a simple manner. The asymptotic expansions for small fields β are

$$-\frac{12\pi}{\alpha}\times Q_{\rm b}|_{y=z=0} = \sum_{k=1}^{\infty}(1-2^{1-2k})\frac{B_{2k}}{2k}\beta^{2k} = \frac{1}{24}\beta^2 - \frac{7}{960}\beta^4 + \frac{31}{8064}\beta^6 + \mathrm{O}(\beta^8) \tag{6.3a}$$

and

$$-\frac{4\pi}{\alpha}\times P_{\rm b}|_{y=z=0} = -4\sum_{k=1}^{\infty}(-1)^k\left(\frac{I_2^k}{k} - \frac{I_1^{k+1}}{k+1}\right)\beta^k = \frac{7}{360}\beta^2 + \frac{33}{20160}\beta^4 + \mathrm{O}(\beta^6) \tag{6.3b}$$

where the I_k^j are rational numbers defined by

$$I_j^k \equiv \int_{-1/2}^{+1/2} \mathrm{d}x\, x^j B_k(x)$$

and the $B_j(x)$ are the Bernoulli polynomials [GD80] with the shorthand that $B_j = B_j(0)$.

High field approximations to all the scalars can be found in the Landau sum representations in that they converge more rapidly the greater the field strength is, but specific expansions in the parameter β^{-1} are more desirable. The high field expansion of Q_b is simply found from the Taylor expansion of the psi function (see Abramowitz and Stegun [AB65], equation 6.4.4) as

$$\begin{aligned}\frac{-12\pi}{\alpha} \times Q_\mathrm{b}|_{y=z=0} &= \ln\beta + \psi(\tfrac{1}{2}) + \sum_{k=1}^{\infty}(-1)^{k+1}(2^{k+1}-1)\beta^{-k}\zeta(k+1) \\ &= \ln\beta - 1.963\,510\,026 + 4.934\,802\,200\,\beta^{-1} + \mathrm{O}(\beta^{-2}) \qquad (6.4a)\end{aligned}$$

which is a convergent sum when $\beta > 2$, and that the leading term is logarithmic. The corresponding expansion for P_b is

$$\begin{aligned}\frac{-4\pi}{\alpha} \times P_\mathrm{b}|_{y=z=0} &= -\tfrac{1}{3} + 4\int_{-1/2}^{+1/2} \mathrm{d}x\, x^2\psi(x+1) + \tfrac{1}{3}\ln\beta \\ &\quad + \beta^{-1}\left\{2 - 4\int_{-1/2}^{+1/2} \mathrm{d}x\, x\psi(x+1)\right\} - 2\beta^{-2} \\ &\quad + 4\sum_{k=3}^{\infty}\frac{(-1)^{k-1}(k-2)}{k(k-1)}\left(2^{k-1}-1\right)\beta^{-k}\zeta(k-1) \\ &= 0.\dot{3}\ln\beta - 0.600\,784\,782\,9 + 1.386\,294\,361\,1\,\beta^{-1} - 2.0\,\beta^{-2} + \mathrm{O}(\beta^{-3}) \qquad (6.4b)\end{aligned}$$

also when $\beta > 2$.

The exact and full dependence of the scalars on the magnetic field strength is shown in figure 2. A few salient points about the behaviour of the scalars with respect to the magnetic field strength should be noted from figure 2 as compared with fermion results in [BK75].

1. All relevant physical quantities depend on Q and P only and R is not independent.

2. All are negative for all field strengths, except for Q_f.

3. In the boson case the order $|R_\mathrm{b}| > |P_\mathrm{b}| > |Q_\mathrm{b}|$ is maintained for all field strengths, but the order changes for the fermion ones. In this case, for low field strengths $|R_\mathrm{f}| > Q_\mathrm{f} > |P_\mathrm{f}|$ but for high fields $Q_\mathrm{f} > |R_\mathrm{f}| > |P_\mathrm{f}|$.

4. The fermion scalars are generally, for low field strengths, some factors larger in magnitude than the corresponding boson scalars and are exactly

$$\frac{Q_\mathrm{f}}{|Q_\mathrm{b}|} \to 8 \qquad \frac{|P_\mathrm{f}|}{|P_\mathrm{b}|} \to \frac{16}{7} \qquad \frac{|R_\mathrm{f}|}{|R_\mathrm{b}|} \to \frac{16}{7} \qquad \text{as } \beta \to 0\,. \qquad (6.5)$$

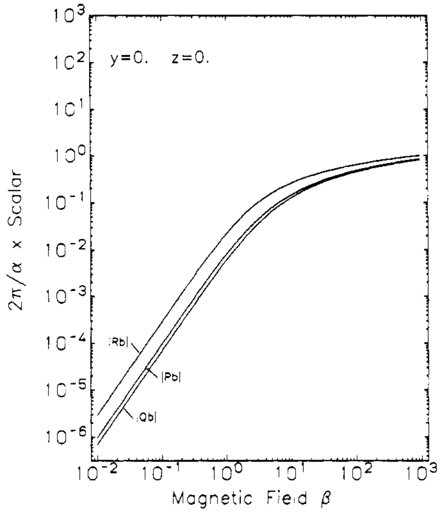

Figure 2. The static and uniform scalars, $|Q_\mathrm{b}|$, $|P_\mathrm{b}|$, and $|R_\mathrm{b}|$ against magnetic field strength β.

5. Q_f rises linearly with increasing field strength in contrast to the other fermion scalars and all the boson scalars, including $|Q_\mathrm{b}|$, which go as $\ln \beta$. This point was made originally in [BK75] with respect to the fermion scalars.

The explanation for points four and five lies with a consideration of the kinematics of virtual processes and with the conservation constraints $\mathcal{E} = \mathcal{E}'$ and $\mathcal{P}_{\|} = \mathcal{P}_{\|}'$ for transitions from one state (unprimed) to another (primed). In the boson case these constraints imply that $n = n'$ whereas in the spin-$\frac{1}{2}$ case it implies $2n + 1 - \sigma = 2n' + 1 - \sigma'$ where σ is the electron spin. Given that the spin is either plus or minus unity then $\Delta\sigma = 0, +2, -2$ and thus one has $\Delta n = 0, +1, -1$ respectively. As the magnetic field strength increases then the Landau energy spacing accordingly increases and the contribution of the energy changing transitions declines, but the spin-flip transitions do not entail any energy change and so dominate in the fermion scalars.

The electric and magnetic polarizabilities (see [WT88] for definitions) can be decomposed into longitudinal and transverse parts like

$$Z_{ij} = Z_\perp \mathrm{I}^\perp_{ij} + Z_{\|}\mathrm{I}^{\|}_{ij} \qquad \Lambda_{ij} = \Lambda_\perp \mathrm{I}^\perp_{ij} + \Lambda_{\|}\mathrm{I}^{\|}_{ij} \tag{6.6}$$

where $\mathrm{I}^{\|}_{ij} = B_i B_j / B^2$ and $\mathrm{I}^\perp_{ij} = \delta_{ij} - \mathrm{I}^{\|}_{ij}$. In the either case the longitudinal and transverse components are then $Z_{\|} = 1 + Q$, $Z_\perp = 1 + P$, $\Lambda_{\|} = 1 + P + \beta\partial P/\partial\beta$ and $\Lambda_\perp = 1 + P$. Point number two implies that for fermions $\delta \boldsymbol{E}_\mathrm{tot}$ is reduced, and to a greater extent, parallel to the external field (dielectric) and is enhanced, to a lesser extent, perpendicular to the field (para-electric). The magnetic perturbations, $\delta \boldsymbol{B}_\mathrm{tot}$, are enhanced in all directions, but more so in the parallel direction (paramagnetic). Again this is not new. In contrast the electric field perturbations in the boson vacuum are enhanced in all directions but more so in the perpendicular direction (para-electric). The magnetic perturbations are also enhanced in all directions (paramagnetic) and more so in the parallel direction.

The explanation for the occurence of dielectric behaviour for the fermion vacuum, in the direction of the external field, as opposed to para-electric screening for the boson case lies in a consideration of vacuum fluctuations. The relevant scalars in this phenomena are $Q_{\rm b}$ and $Q_{\rm f}$ which have the following Landau sum representations:

$$Q_{\rm b} \sim \sum_{n=0}^{\infty} \left\{ (n+\tfrac{1}{2}+\beta^{-1})^{-1} - \ln\left[\frac{n+1+\beta^{-1}}{n+\beta^{-1}}\right] \right\} \tag{6.7a}$$

$$Q_{\rm f} \sim \sum_{n=0}^{\infty} \left\{ \tfrac{1}{2}(n+\beta^{-1})^{-1} + \tfrac{1}{2}(n+1+\beta^{-1})^{-1} - \ln\left[\frac{n+1+\beta^{-1}}{n+\beta^{-1}}\right] \right\}. \tag{6.7b}$$

As can be seen from these equations the fermion vacuum fluctuations involve contributions from both Landau levels n and $n+1$, whereas the boson case only involves one level at n. This level is situated exactly midway between the two fermion levels because of the $\frac{1}{2}$ term in $Q_{\rm b}$, reflecting the zero-point contribution. It should be noted that the renormalizing term is identical in both cases. Now it is generally true that

$$\tfrac{1}{2}x^{-1} + \tfrac{1}{2}(x+1)^{-1} - \ln\left(\frac{x+1}{x}\right) = \int_0^1 {\rm d}t \frac{(t-\frac{1}{2})^2}{(x+1-t)^2(x+t)^2} > 0$$

for all real, positive x. Thus the fermion scalar is always positive. It is also generally true that

$$(\tfrac{1}{2}+x)^{-1} - \ln\left(\frac{x+1}{x}\right) = -(x+\tfrac{1}{2})^{-1} \int_0^1 {\rm d}t \frac{(t-\frac{1}{2})^2}{(x+1-t)(x+t)} < 0$$

for all real x greater than $+\frac{1}{2}$. The boson scalar is then always negative. Thus it can be seen that the difference arises from the two distinct types of eigenvalue spectra and spin degeneracies the two particles possess.

The static longitudinal dielectric function is simply given by

$$\varepsilon_{\parallel}(q, \omega = 0) = 1 + q^{-2}\left(q_\perp^2 P + q_\parallel^2 Q\right) \tag{6.8}$$

for either the boson or fermion cases. In order to study the effect of the external magnetic field on the screened electrostatic potential of a test charge it is useful to calculate the high field limits, because the greatest difference with a bare Coloumb form is expected in this regime. The appropriate expressions for the scalars are then found from equations (4.2) and others and by considering only terms up to orders of β^{-1} one has

$$\varepsilon_{\parallel}^{\rm b} = 1 + \frac{\alpha}{4\pi q^2}\left\{ -\tfrac{1}{3}q^2 \ln\beta + \tfrac{1}{3}q_\perp^2 - 4q_\perp^2 \int_{-1/2}^{+1/2} {\rm d}x\, x^2 \psi(x) - \tfrac{1}{3}q_\parallel^2 \psi(\tfrac{1}{2}) + {\rm O}(\beta^{-1}) \right\}. \tag{6.9}$$

The screened potential, as a function of the parallel distance r and the perpendicular distance ρ from the central charge, is found from the static dielectric function in the usual way (see [WT88]). The screening due to the boson vacum in the high-field limit is then

$$V_\star^{\rm b} = \frac{1}{a_\perp}\left(r^2 + \frac{a_\parallel}{a_\perp}\rho^2\right)^{-1/2} \tag{6.10}$$

which shows an overall enhancement as well as an anisotropy relative to the field direction. The constants involved here are

$$a_{\parallel} = 1 + \frac{\alpha}{12\pi}(-\ln\beta + \gamma + 2\ln 2) \tag{6.11a}$$

$$a_{\perp} = 1 + \frac{\alpha}{12\pi}\left(-\ln\beta + \gamma - 12 + 12\ln 3 + 3\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{2k+3}2^{-2k}\right) \tag{6.11b}$$

and for all reasonable field strengths these will be positive and $\gamma = -\psi(1)$ also.

In the fermion case the dielectric function has a quite different form:

$$\varepsilon_{\parallel}^{\mathrm{f}} = 1 + \frac{\alpha}{\pi q^2}\beta\left[1 - |q_{\parallel}|^{-1}(1+\tfrac{1}{4}q_{\parallel}^2)^{-1/2}\ln\left(\frac{\sqrt{4+q_{\parallel}^2}+|q_{\parallel}|}{\sqrt{4+q_{\parallel}^2}-|q_{\parallel}|}\right)\right] - \frac{\alpha}{3\pi}\ln\beta + \frac{\alpha}{6\pi q^2}(5q_{\perp}^2 + 2\gamma q_{\parallel}^2) + \mathrm{O}(\beta^{-1}) \tag{6.12}$$

where the ψ functions have been expanded in a descending series in β. The leading-order term is linear in the field strength with a different dependence on the wavenumber. As a consequence at large radial distances from the central charge s, the resulting potential is an anisotropic Coloumb type

$$V_{\star}^{\mathrm{f}} \sim \left[r^2 + \rho^2\left(1 + \frac{\alpha\beta}{6\pi}\right)\right]^{-1/2} \tag{6.13a}$$

while at small distances one has a finite-range isotropic potential

$$V_{\star}^{\mathrm{f}} \sim s^{-1}\mathrm{e}^{-(\alpha\beta/\pi)^{1/2}s}\,. \tag{6.13b}$$

7. Analytic structure of scalars

The dependence of the scalars on z, for real $z > 0$, with y fixed is not necessarily a monotonic one but for sufficiently large z the magnitude of the scalar will decrease exponentially with increasing z. The basic singular behaviour of the scalars on y for a given, arbitrary value of z can be found from the dispersion sum representations by examining the argument of the psi functions, which can be written most generally as

$$\beta^{-1}\left[1 + y(\tfrac{1}{4} - x^2)\right] - \tfrac{1}{2}(p+q+a) + x(q-p+b) + n + \tfrac{3}{2} \tag{7.1}$$

with integers a and b appropriate to the particular scalar. The pair threshold values of y are then given by one of the roots to an additional quadratic equation which expresses the condition that the above quadratic in x, equation (7.1), has coincident real roots. In summary we have the following.

1. $\mathrm{Re}(\overline{Q}_{\mathrm{b}})$ has inverse square-root singularities on the upper sides of the pair creation thresholds, $y = -\{\mathcal{E}_i + \mathcal{E}_j\}^2$ for $i \neq j \geq 0$ and $z \neq 0$, but only cusps at $y = -\{\mathcal{E}_i + \mathcal{E}_i\}^2$, $i \geq 0$ for all z (including $z = 0$). Approaching the threshold from above $\mathrm{Re}(\overline{Q}_{\mathrm{b}}) \to +\infty$. $\mathrm{Im}(\overline{Q}_{\mathrm{b}})$ has inverse square-root singularities on the lower sides

of the above thresholds, under the same conditions, and cusps at the same points. The other difference is that the cusps are downwards pointing.

2. $\mathrm{Re}(\overline{Q}_f)$ has inverse square-root singularities on the upper sides of the thresholds $y = -\{\mathcal{E}_i + \mathcal{E}_j\}^2$ for $i, j \geq 0$ when $z \neq 0$, but only ones at $y = -\{\mathcal{E}_i + \mathcal{E}_i\}^2$ for $i \geq 0$ if $z = 0$. $\mathrm{Re}(\overline{Q}_f) \to +\infty$ as the threshold is approached from above. $\mathrm{Im}(\overline{Q}_f)$ has inverse square-root singularities on the lower sides of the same thresholds, and under the same conditions. Again $\mathrm{Im}(\overline{Q}_f) \to +\infty$ on the lower side of the thresholds. Here it is understood that the appropriate energy formula applies in the two spin cases even though the same symbol has been used for both.

For the exact behaviour of the other boson and fermion scalars the reader is referred to [WT89]. The dependence of both the real and imaginary parts of two of the scalars, Q_b and Q_f, on both the kinematic invariants, y and z, is illustrated in figures 3 and 4 respectively.

8. Eigenmodes

The exact solutions to the dispersion equations (1.7) are the full complex solutions $y_r + iy_i$ as functions of $z_r + iz_i$ with the complete Hermitian and anti-Hermitian parts of the scalars included, i.e. $S_r + iS_i$ for general scalars S. The scalars have real and imaginary parts even if the photon parameters y and z are purely real. The imaginary part of y comes from the imaginary part of the frequency $\Omega = \Omega_r - i\Gamma$ where $\Gamma > 0$ is half the exact inverse lifetime and the real part is the physical propagation frequency. The imaginary part of z is found [SH75] to have many solutions (an infinite number) and is related to the principal internal quantum number of the quasibound positronium state, while the real part takes the standard interpretation. What is generally assumed is that the damping time constant Γ is much smaller than the real part and that z is purely real. The consequence of this is that equations (5.1) and (5.2) holds true for the inverse lifetime and the approximate real solutions are given by equations (1.7) with only the Hermitian parts of the scalars involved.

Because the scalars have two complex valued branches there are two different sets of dispersion relations, and the set corresponding to the real and imaginary parts given in this paper will be denoted the physical branch. The second branch is always related to the first by

$$\mathrm{S}(y - i0) - \mathrm{S}(y + i0) = 2i\,\mathrm{Im}\,\mathrm{S} \tag{8.1}$$

and close to the pair production thresholds the choice of the sheet in the dispersion relations corresponds to the choice of a different sign of the square root in the momentum integrals, in the Landau sum representation. In this section only the existence and types of real solutions to the dispersion relations, equations (1.7), on the principal branch will be discussed. The distinct fermion and boson modes described by the dispersion relations of modes 1, 2, and 3 in equations (1.7) are hereafter denoted by the symbols F1, F2, F3 and B1, B2, B3 respectively.

For all β (less than the ultrastrong field strength $\sim e^{2\pi/\alpha}$) P_b and P_f are never less than -1 so that no solutions on the principal branch exist for mode 1 in the boson or fermion case as can be seen from the following argument: they have local, finite minima on the lower side of the pair thresholds at $z = 0$ and it would be at these points where P_b or P_f would equal -1 first, for some field strength and particular

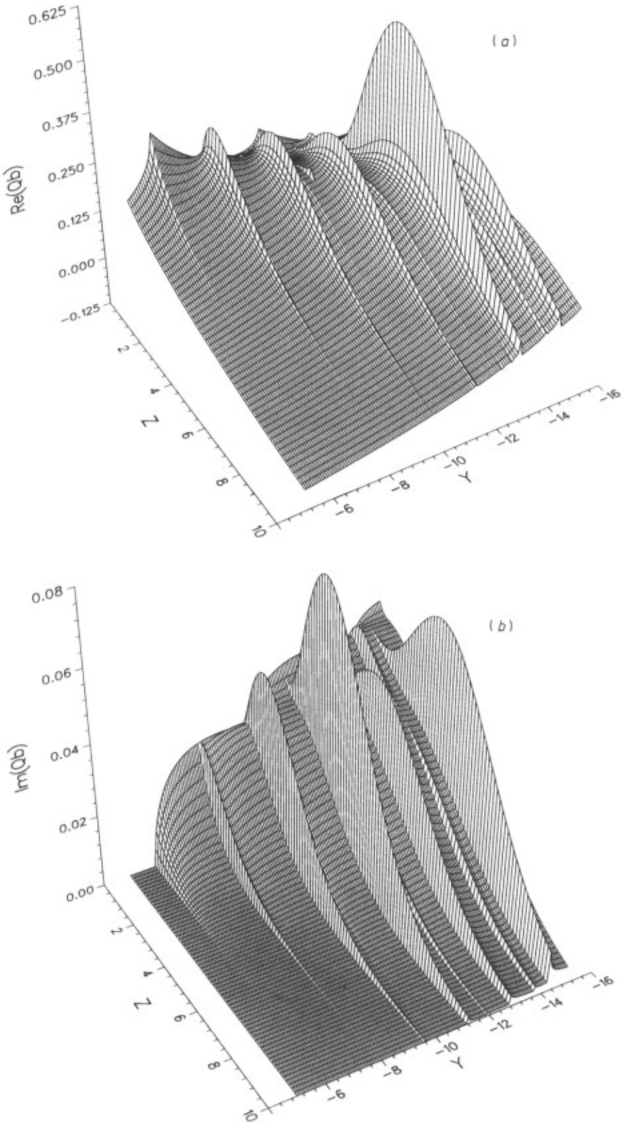

Figure 3. (a) **The real part of Q_b against y and z for $\beta = 1.0$.** (b) **The imaginary part of Q_b against y and z for $\beta = 1.0$.**

threshold, if it were possible. However the magnitude of the P scalars at these points are only weakly increasing with field strength and the Landau level number of the threshold. This conclusion only holds rigorously at $z = 0$.

The situation for the modes 2 and 3 can be conveniently discussed by looking at the approximate dispersion solutions in the neighbourhood of the pair production thresholds, as was done in the fermion case in [SH75]. As mentioned earlier it was first noted in this paper and [SH72] that the dispersion solutions for modes 2 and 3 in the fermion case departed from the lightcone trajectory and flattened out along the upper edge of sets of pair production thresholds. This was established by the discovery of these approximate solutions and an attempt at painting a global picture

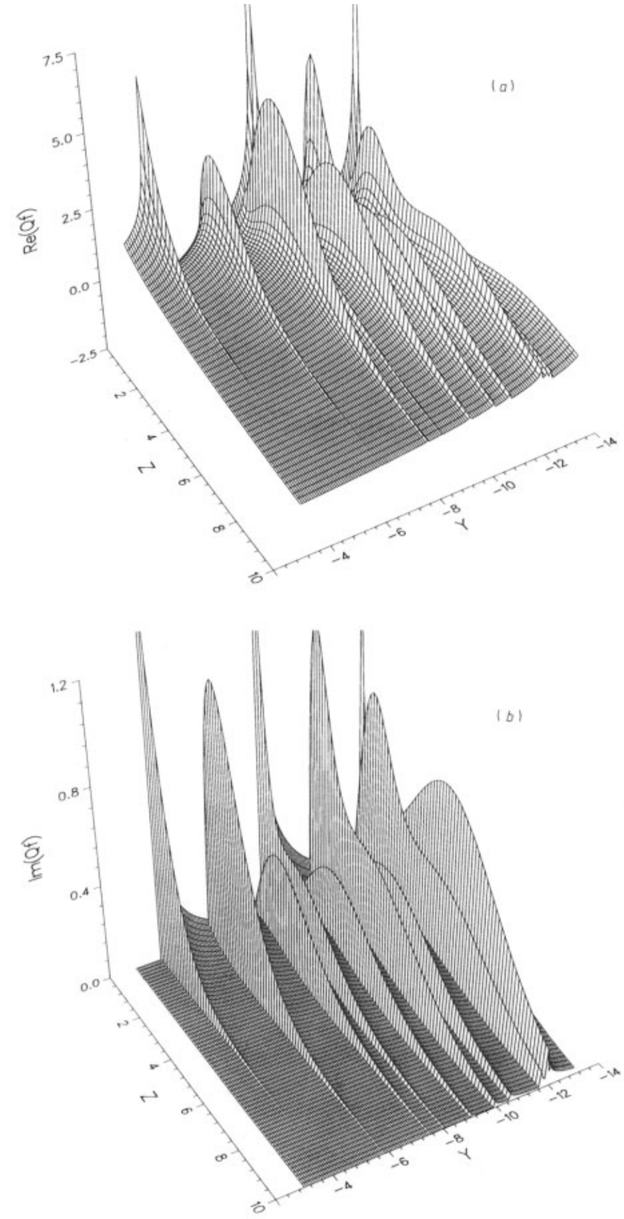

Figure 4. (*a*) The real part of Q_f against y and z for $\beta = 1.0$. (*b*) The imaginary part of Q_f against y and z for $\beta = 1.0$.

of the modes was made by patching the results at the thresholds together. Near a particular threshold, say $y_{n,n'}$, only that term in the Landau sum with the appropriate momentum integral need be retained as this will be the dominant contribution (the contact term for this (n, n') pair is also dropped). Also taking the lowest order term in an expansion of the momentum integral about $y - y_{n,n'} = 0$ yields an inverse square root dependence on y. For example the mode F2 near the highest threshold $(0, 0)$, is governed by the equation

$$y + z = \alpha\beta e^{-z}\left[y + 4\right]^{-1/2} . \tag{8.2}$$

Using the above approximations for mode B2 one finds a remarkable cancellation of the singular terms between $\overline{Q}_{\rm b}$ and $\overline{P}_{\rm b}$ so that the dispersion equation is

$$y \;+\; z \;+\; \alpha O(1) \;= 0 \tag{8.3}$$

and thus there is only slight deviation from the lightcone, for all values of n and n'. Clearly this mode, which is of a mixed transverse and longitudinal nature, is virtually a pure photon eigenstate and experiences no mixing with bound boson–antiboson pairs. The situation with regard to mode B3 is different, however, and one can verify that the approximate relation is

$$y + z = \tfrac{1}{2}\alpha\beta^2 e^{-z}(\mathcal{E}_{n'}\mathcal{E}_n)^{-1/2}[\Theta^-_{n',n}(z)]^2[y + \{\mathcal{E}_{n'}+\mathcal{E}_n\}^2]^{-1/2} \;+\; \alpha O(1)\,. \tag{8.4}$$

This has a basically similar set of solutions as mode F2 except in one interesting regard. The factor $\Theta^-_{n',n}$ in equation (8.7) can null for certain real, non-negative values of z and thus the solution will cross (or touch) the pair threshold exactly at this value of transverse wavenumber.

Some of the solutions to the dispersion relations of the magnetized boson and fermion vacua have been numerically found to illustrate the points made in the preceeding subsection and elsewhere. Some of these are shown in figures 5–7. The modes taken are B3, F2 and F3 in the regions of the two highest thresholds. The B2 mode is absent because the deviations from the free-space dispersion solutions are quite small everywhere. It should be reported that the [CV74, BK76] numerical solution of the 'longitudinal massive photon' has been confirmed by the author and the dependence of the cut-off frequency of this mode (at $q_{||} = 0$) on the magnetic field strength is correct as they have found. However, there is no point in presenting them again.

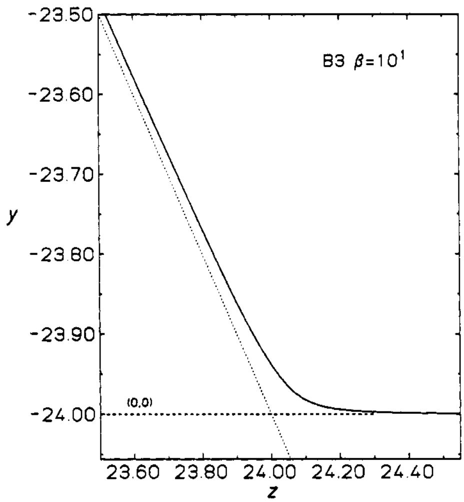

Figure 5. Dispersion solutions for mode B3 above the (0,0) threshold for $\beta = 10^1$

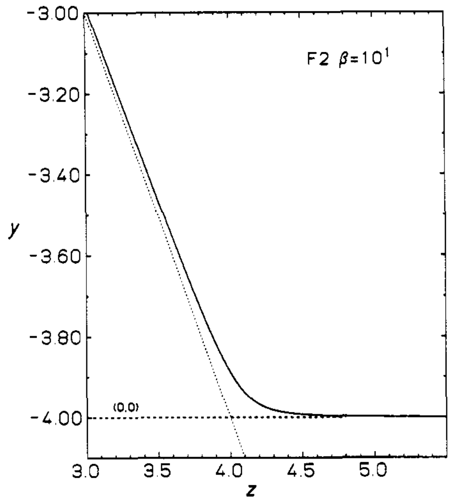

Figure 6. Dispersion solutions for mode F2 above the (0,0) threshold for $\beta = 10^1$

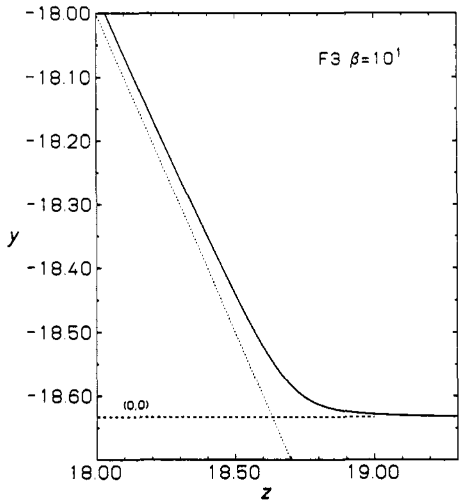

Figure 7. Dispersion solutions for mode F3 above the (0,0) threshold for $\beta = 10^1$

9. Conclusion

In this paper the polarization tensor for the magnetized bosonic vacuum, calculated in the random phase approximation to the equations of motion of current fluctuations in scalar quantum electrodynamics, has been studied. This result has been presented in three different representation forms: the proper-time, the spectral and the dispersion sum forms. The imaginary parts have also been explicitly given along with the approximate inverse lifetimes. The static and uniform properties of both vacua have been compared and it is found that the bosonic vacuum acts as a para-electric medium and the parallel electric polarizability only has a logarithmic large field growth. The behaviour of both the bosonic and fermion scalars of the photon momenta is extensively explored numerically where it is found that the boson scalar $Q_{\rm b}$ does not exhibit any of the inverse square root singularities where the creation of virtual boson–antiboson pairs is first allowed. A consequence of these two results is that the 'massive longitudinal photon' is not supported in the bosonic vacuum state.

Furthermore analytic and numerical solution of the fermion and boson dispersion relations confirms the earlier investigations in the former case and another difference with the latter. In the bosonic vacuum only one purely transverse electromagnetic mode (mode 3) acquires mass through mixing with quasibound boson–antiboson pairs and becomes channelled along the external magnetic field lines. The other mode, whose electric polarization vector can acquire some components parallel to the wavevector of propagation, (mode 2) only weakly departs from the free-space dispersion law and is not deflected at the pair production thresholds.

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